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SOME GENERALIZED TYPE FUNCTIONAL DEPENDENCIES FORMALIZED AS EQUALITY SET ON MATRICES

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In this paper we deal with functional (f) dependencies and their generalizations, the dual, strong (d, s) and weak (w) dependencies. We give new axioms instead of Armstrong's and Czédli's and axiomatize the full w-families. Our axioms are based on a characterization of equality-sets of matrices. We demonstrate an essential difference between the weak dependency and the rest. We give an estimation for the number of rows which is needed for some antichain of an n-element set to represent it as the set of candidate keys in a relation. Finally, we give a combinatorial characterization of the generating sets with minimal cardinality of full f-families.

0. Introduction

According to E.F. Codd [6] a relation is a matrix without two identical rows. Rows correspond to data records and columns to the attributes that are to be stored of a data item. He also introduced [7] the concept of *functional dependency:* a set of columns depends on another if fixing the values in a row taken on the first determine those on the second.

Other concepts of his are the *key* (a set of attributes on which all depend) and the *candidate key* (a minimal key).

Candidate keys clearly do not contain each other [12].

The possible mathematical structure of functional dependencies was first investigated by W.W. Armstrong [1]. Among others he found that this structure is determined by the *maximal dependencies* (those which have maximal attribute subsets depending on minimal ones) and even by the dependent sides of the maximal dependencies. We also heavily use these 'maximal dependent subsets of attributes' as technical tools.

Different kinds of functional dependency have also been introduced [3, 9, 13, 15] and axiomatized, usually in similar systems to Armstrong's [8]. [10] discusses an interesting connection between the decomposition of relational data bases and the boolean switching functions.

The harder problems of the topic are usually of combinatorial nature (see [4, 5, 11, 16]).

In this paper in Section 1 we give the formal definition of the functional, dual, strong and weak dependencies and give new axioms for full f-, d- and s-families.

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In Section 2 develop the analogy and differences among the dependencies of different types and give an axiom for full w-families.

In Section 3 we deal with a question stated in [11].

Certain dependencies of a relational data base are known by its designer. We call these *initial dependencies*. In general initial dependencies imply new dependencies. W.W. Armstrong [2] has developed a method to find the dependencies implied by a given set of initial functional dependencies. He also gave a characterization of the sets of initial dependencies that imply all the dependencies of a given full f-family and are of minimal cardinality. This characterization has a logical nature; we give a combinatorial equivalent of it.

We use the following notational conventions:

Ω denotes the set of attributes, $P(\Omega)$ denotes his power set. If g is a function with X as its domain and $Z \subseteq X$, then $g\restriction_Z$ denotes the function which has domain Z and for any $z \in Z$, $g(z) = g\restriction_Z(z)$. $\subset$ means strict inclusion.

1. Old and new axioms

We start with the definitions of functional, dual, strong and weak dependencies based on [1] and [8].

Definition 1.1. Let A, B be subsets of Ω and let R be a relation over Ω. Then we say that B (i) *functionally;* (ii) *dually;* (iii) *strongly;* (iv) *weakly depends on A in R* if

(i) $(\forall g, h \in R)\quad (g\restriction_A = h\restriction_A \rightarrow g\restriction_B = h\restriction_B)$;

(ii) $(\forall g, h \in R)\quad ((\exists a \in A)(g(a) = h(a)) \rightarrow (\exists b \in B)(g(b) = h(b)))$;

(iii) $(\forall g, h \in R)\quad ((\exists a \in A)(g(a) = h(a)) \rightarrow g\restriction_B = h\restriction_B)$;

(iv) $(\forall g, h \in R)\quad (g\restriction_A = h\restriction_A \rightarrow (\exists b \in B)(g(b) = (h(b)))$

holds respectively and denote these by

$$A \xrightarrow[R]{\mathrm{f}} B, \qquad A \xrightarrow[R]{\mathrm{d}} B, \qquad A \xrightarrow[R]{\mathrm{s}} B, \qquad A \xrightarrow[R]{\mathrm{w}} B$$

corresponding to the type of the denoted dependency.

Example. Let Ω = {author, title, hall, shelf}. Let we have a library with eighteen books, three halls and three shelves in every hall; one shelf holds two books. Let the relation R containing the data of the library be given by Table 1. Thus

$$\{\text{author, title}\} \xrightarrow[R]{\mathrm{d}} \{\text{hall, shelf}\}$$

holds, and for $i = 1, \ldots, 12$ the book by author i and entitled i is on the $(1 + 3 \cdot \{\frac{1}{3}i\})$-th hall ($[x]$ denotes the whole part and $\{x\}$ the fraction part of x). The reader, knowing

Table 1

author	title	hall	shelf
1	1	1	2
2	2	1	3
3	3	1	1
4	4	1	2
5	5	2	3
6	6	2	1
7	7	2	2
8	8	2	3
9	9	3	1
10	10	3	2
11	11	3	3
12	12	3	1
1	4	1	1
5	8	3	3
4	1	1	3
7	10	3	2
6	10	2	2
6	9	2	1

the author or the title of the required book, may find it without examining the whole library: for example if i is the author of the book, then it is enough to look the $[\frac{1}{4}(i+3)]$-th hall, and the $(1+3\cdot\{\frac{1}{3}i\})$-th shelves of the other two halls.

In R {author, title} $\xrightarrow[R]{\mathrm{f}}$ {hall, shelf} holds too, but to store this functional dependency is equivalent to store the table of R; the {author, title} $\xrightarrow[R]{\mathrm{d}}$ {hall, shelf} dependency is more effective.

For proving the effectiveness of these dependencies we elaborated in the Automation Institute of the Hungarian Academy of Sciences a large-sized practical application of the relational data model.

We have planned an inventory-recording system for an agricultural corporation. The task of the system is to organize the component-traffic of about 350 agricultural estates. More exactly the task is: to record the inventory-stores, the orders of customers, to help the decisions making in this field and to help services.

First we used a traditional system concept for this purpose. Later this concept was transformed into the relational data model based on recent investigations. We saved about 40 percent of the memory capacity in this way. With using the results of Aho, Sagiv and Ullmann about relational expressions, we proved that the response time remained in the same order.

If R is a relation over Ω, and $Y\in\{F,D,S,W\}$ and $y\in\{\mathrm{f},\mathrm{d},\mathrm{s},\mathrm{w}\}$ corresponds to Y, then we use the notation

$$Y_R=\{(A,B)\colon A\xrightarrow[R]{y}B\}.$$

We call *full y-families* the sets having this form.

In order to investigate the various dependencies the first step is the axiomatization of full y-families for $y \in \{\mathrm{f}, \mathrm{d}, \mathrm{s}, \mathrm{w}\}$. In [1] there is a system of axioms for full f-family and in [8] there are for full d- and s-families. For the sake of completeness we reproduce them here.

Let $Y \subseteq P(\Omega) \times P(\Omega)$. Then we say that Y satisfies the f-axioms, if for all $A, B, C, D \subseteq \Omega$

(F1) $(A, A) \in Y$;

(F2) $(A, B) \in Y,\ (B, C) \in Y \rightarrow (A, C) \in Y$;

(F3) $(A, B) \in Y,\ A \subseteq C,\ D \subseteq B \rightarrow (C, D) \in Y$;

(F4) $(A, B) \in Y,\ (C, D) \in Y \rightarrow (A \cup C, B \cup D) \in Y$.

Y satisfies the ϑ-axioms if for all $A, B, C, D \subseteq \Omega$

(D1) $(A, A) \in Y$;

(D2) $(A, B) \in Y,\ (B, C) \rightarrow (A, C) \in Y$;

(D3) $(A, B) \in Y,\ C \subseteq A,\ B \subseteq D \rightarrow (C, D) \in Y$;

(D4) $(A, B) \in Y,\ (C, D) \in Y \rightarrow (A \cup C, B \cup D) \in Y$;

(D5) $(A, \emptyset) \in Y \rightarrow A = \emptyset$.

Y satisfies the γ-axioms if for all $A, B, C, D \subseteq \Omega$

(S1) $(\{a\}, \{a\}) \in Y$;

(S2) $(A, B) \in Y,\ (B, C) \in Y,\ B \neq \emptyset \rightarrow (A, C) \in Y$;

(S3) $(A, B) \in Y,\ C \subseteq A,\ D \subseteq B \rightarrow (C, D) \in Y$;

(S4) $(A, B) \in Y,\ (C, D) \in Y \rightarrow (A \cap C, B \cup D) \in Y$;

(S5) $(A, B) \in Y,\ (C, D) \in Y \rightarrow (A \cup C, B \cap D) \in Y$.

We need the following technical lemma.

Lemma 1.1. *Let $F \subseteq P(\Omega) \times (\Omega)$ be such that $(X, Y) \in F$ and $Y \neq \emptyset$ imply $X \neq \emptyset$. Then F satisfies the f-axioms iff $D = \{(A, B) : (B, A) \in F\}$ satisfies the ϑ-axioms.*

Proof. Trivial by the f- and ϑ-axioms. (D5) makes necessary the assumption that $(X, Y) \in F$ and $Y \neq \emptyset$ imply $X \neq \emptyset$. $\square$

Remark. The assumption ($(X, Y) \in F$ and $Y \neq \emptyset$ imply $X \neq \emptyset$) in Lemma 1.1 is not an important restriction: if F satisfies the f-axioms let $F' = F \setminus \{(\emptyset, X) : X \neq \emptyset\}$. Then F' obviously satisfies the f-axioms and the critical assumption as well and we have: $X \neq \emptyset$ implies that $(X, Y) \in F \leftrightarrow (X, Y) \in F'$.

In the following we give new axioms instead of the f-, ϑ- and γ-axioms and give an axiom that characterizes the *weak full w-families* which is such a full w-family that whenever (X, Y) is an element of the family then X is not void.

F-axiom. Let $F \subseteq P(\Omega) \times P(\Omega)$. Then we say that *$F$ satisfies the F-axiom* if for any $(X, Y) \in P(\Omega) \times P(\Omega) \setminus F$ there is an $E \subseteq \Omega$ such that

(i) $X \subseteq E$ and $Y \not\subseteq E$;

(ii) if $(X', Y') \in F$ and $X' \subseteq E$, then $Y' \subseteq E$.

D-axiom. Let $D \subseteq P(\Omega) \times P(\Omega)$. Then we say that *$D$ satisfies the D-axiom* if for any $(X, Y) \in P(\Omega) \times P(\Omega) \setminus D$ there is an $E \subseteq \Omega$ such that

(i) $X \cap E \neq \emptyset$ and $Y \cap E = \emptyset$;

(ii) if $(X', Y') \in D$ and $X' \cap E \neq \emptyset$, then $Y' \cap E \neq \emptyset$.

S-axiom. Let $S \subseteq P(\Omega) \times P(\Omega)$. Then we say that *$S$ satisfies the S-axiom* if for any $(X, Y) \in P(\Omega) \times P(\Omega) \setminus S$ there is an $E \subseteq \Omega$ such that

(i) $X \cap E \neq \emptyset$ and $Y \not\subseteq E$;

(ii) if $(X', Y') \in S$ and $X' \cap E \neq \emptyset$, then $Y' \subseteq E$.

W-axiom. Let $W \subseteq P(\Omega) \times P(\Omega)$. Then we say that *$W$ satisfies the W-axiom* if for any $(X, Y) \in P(\Omega) \times P(\Omega) \setminus W$ there is an $E \subseteq \Omega$ such that

(i) $X \subseteq E$ and $Y \cap E = \emptyset$;

(ii) if $(X', Y') \in W$ and $X' \subseteq E$, then $Y' \cap E \neq \emptyset$.

Theorem 1.1. (i) *Let $F \subseteq P(\Omega) \times P(\Omega)$. Then F satisfies the f-axioms iff F satisfies the F-axiom.*

(ii) *Let $D \subseteq P(\Omega) \times P(\Omega)$. Then D satisfies the ϑ-axioms iff D satisfies the D-axiom.*

(iii) *Let $S \subseteq P(\Omega) \times P(\Omega)$. Then S satisfies the γ-axioms iff S satisfies the S-axiom.*

Proof. (i) Suppose that F satisfies the F-axiom. Then

(F1) If $(A, A) \notin F$, then there is an $E \subseteq \Omega$ such that $A \subseteq E$ and $A \not\subseteq E$, which is a contradiction.

(F2) If $(A, B) \in F$, $(B, C) \in F$ and $(A, C) \notin F$, then there is an $E \subseteq \Omega$ such that $A \subseteq E$ and $C \not\subseteq E$. Furthermore $(A, B) \in F$, $A \subseteq E$ imply $B \subseteq E$ and using $(B, C) \in F$, $C \subseteq E$ which is a contradiction.

The proof of (F3) and (F4) is analogous.

Suppose now that F satisfies the f-axioms. Let $(A, B) \in P(\Omega) \times P(\Omega) \setminus F$.

Claim. There is an $E \subseteq A$ such that $(E, B) \in P(\Omega) \times P(\Omega) \setminus F$ and $E' \supset E$ implies $(E', B) \in F$.

$(\Omega, \Omega) \in F$ by (F1). Thus, by (F3) $(\Omega, B) \in F$ holds. $A \subseteq \Omega$ and $(A, B) \in P(\Omega) \times P(\Omega) \setminus F$, consequently there is an $E \subseteq \Omega$ which is maximal w.r.t. the properties $(E, B) \in F$ and $E \supseteq A$.

This E clearly satisfies the restriction of the Claim. Let $E \supseteq A$ which is guaranteed by the Claim. We state that E satisfies (i) and (ii) of the F-axiom. Namely by the choice of E, $A \subseteq E$ holds. By (F1) and (F3) $B \subseteq E$ implies $(E, B) \in F$. Thus we have $B \nsubseteq E$.

Let $(C, D) \in F$ and $C \subseteq E$. $D \nsubseteq E$ implies $E' = D \cup E \supset E$ and by the maximality of E $(E', B) \in F$ holds.

$(E, E) \in F$ by (F1), hence (F4) implies that $(E, E') \in F$. Now $(E, E') \in F$ and $(E', B) \in F$ and (F2) imply that $(E, B) \in F$ which is a contradiction.

(ii) Let $F = \{(A, B) : (B, A) \in D\}$. Then by Lemma 1.1, F satisfies the f-axioms iff D satisfies the ϑ-axioms. Hence, by (i), it is enough to show that F satisfies the F-axiom iff D satisfies the D-axiom.

Suppose that F satisfies the F-axiom. For $(A, B) \in P(\Omega) \times P(\Omega) \setminus F$ let $E(A, B)$ be such a subset of Ω that $A \subseteq E(A, B)$, $B \nsubseteq E(A, B)$ and if both $(A', B') \in F$ and $A' \subseteq E(A, B)$, then $B' \subseteq E(A, B)$. By the F-axiom such an $E(A, B)$ exists. By the definition of F whenever $(A, B) \in P(\Omega) \times P(\Omega)$, then $(A, B) \in P(\Omega) \times P(\Omega) \setminus F$ iff $(B, A) \in P(\Omega) \times P(\Omega) \setminus D$.

Now it is easy to check that for $(B, A) \in P(\Omega) \times P(\Omega) \setminus D$, $\Omega \setminus E(A, B)$ satisfies the D-axiom.

If D satisfies the D-axiom, then F satisfies the F-axiom; this can be shown by the same argument.

(iii) Suppose that S satisfies the S-axiom. Then the proof of the fact that S satisfies the γ-axioms is an easy modification of the proof of (i).

Suppose now that S satisfies the γ-axioms. Let $(A, B) \in P(\Omega) \times P(\Omega) \setminus S$.

Claim. There is an $a \in A$ and an $E \subseteq \Omega$, such that

(a) $a \in E$;

(b) $(\{a\}, E) \in S$; and

(c) $E' \supset E$ implies that $(\{a\}, E') \notin S$.

If for any $a \in A$ we have $(\{a\}, B) \in S$, then $(A, B) \in S$ by the repeated application of (S5). Hence there is an $a \in A$ such that $(\{a\}, B) \notin S$. Now if, for every $b \in B$, $(\{a\}, \{b\}) \in S$ holds, then by the repeated application of (S4) we have $(\{a\}, B) \in S$. Thus there is a $b \in B$ such that $(\{a\}, \{b\}) \notin S$.

By (S1) and (S3) there is an $E \subseteq \Omega$ such that $a \in E$, $(\{a\}, E) \in S$ and E is maximal w.r.t. this property. This E is appropriate for the Claim.

Let $E \subseteq \Omega$ and $a \in A$ guaranteed by the Claim. Then by (S3) we have $b \notin E$. Hence $A \cap E \neq \emptyset$ and $B \cap (\Omega \setminus E) \neq \emptyset$. Now let $(C, D) \in S$ such that $C \cap E \neq \emptyset$; let $c \in C \cap E$. Suppose that $D \cap (\Omega \setminus E) \neq \emptyset$; let $d \in D \cap (\Omega \setminus E)$. By (S3) we have $(\{c\}, \{d\}) \in S$ and by (S1) we have $(\{c\}, \{c\}) \in S$. $(\{a\}, E) \in S$ implies that $(\{a, c\}, \{c\}) \in S$, by (S5). Hence (S3) implies that $(\{a\}, \{c\}) \in S$. Now $(\{a\}, \{c\}) \in S$, $(\{c\}, \{d\}) \in S$ and (S2) imply that $(\{a\}, \{d\}) \in S$. Thus by (S4) we have $(\{a\}, E \cup \{d\}) \in S$ which is a contradiction as $E' = E \cup \{d\} \supset E$.

Consequently the E guaranteed by the Claim demonstrates that S satisfies the S-axiom. $\square$

It is worth to remark how the full y-family can be found (for $y \in \{\mathrm{f}, \mathrm{d}, \mathrm{s}\}$) generated by a given subset of $P(\Omega) \times P(\Omega)$ based on the Y-axiom. Let e.g. $y = \mathrm{f}$ and let be given an $F' \subseteq P(\Omega) \times P(\Omega)$. Then the least full f-family containing F' is the following:

$$F = \{(A,B) : A, B \subseteq \Omega \,\&\, (\forall E \subseteq \Omega)(A \subseteq \mathrm{E} \,\&\, B \not\subseteq E) \rightarrow (\exists (A', B') \in F')(A' \subseteq E \,\&\, B' \not\subseteq E)\}.$$

2. The equality set

Definition 2.1. Let R be a relation over Ω. We define *the equality set of* R, $\mathscr{E}_R$ as follows. For $h, g \in R$ let $E(h,g) = \{a \in \Omega : h(a) = g(a)\}$ and let $\mathscr{E}_R = \{E(h,g) : h, g \in R$ and $h \neq g\}$.

Definition 2.2. Let $\mathbf{A}$ be a set system. Then $\mathbf{A}$ *is a Δ-system* if for any $A, B, C, D \in \mathbf{A}$, $A \neq B$ and $C \neq D$ imply that $A \cap B = C \cap D$.

Remark: It is easy to see that $\mathbf{A}$ is a Δ-system iff for any $A, B \in \mathbf{A}$, $A \neq B$ implies that $A \cap B = \bigcap \mathbf{A}$.

Theorem 2.1. (i) *Let R be a relation over Ω and let h, f, g be different elements of R. Then $E(h,g)$, $E(h,f)$, $E(g,f)$ form a Δ-system.*

(ii) *Let $\mathscr{E} = \{E_{i,j} : 1 \leq i < j \leq k\}$ such that for each $1 \leq i < j < l \leq k$ $\{E_{i,j}, E_{i,l}, E_{j,l}\}$ is a Δ-system. Then there is a relation R over Ω with $\mathscr{E}_R = \mathscr{E}$.*

Proof: (i) By symmetry it is enough to prove that $a \in E(h,g) \cap E(h,f)$ implies $a \in E(g,f)$. But this is trivial as $a \in E(h,g) \cap E(h,f)$ means both $h(a) = g(a)$ and $h(a) = f(a)$. Hence $g(a) = f(a)$, i.e. $a \in E(g,f)$.

(ii) Let $R = \{h_1, \ldots, h_k\}$ where $h_i = ([i], [i], \ldots, [i])$ (every attribute has value $[i]$ now to be explained). The values of attribute a are attribute-specific equivalence classes of integers from 1 to k, where $m \sim n$ on attribute a iff $m = n$ or $a \in E_{m,n}$ (if $m < n$) or $a \in E_{n,m}$ (if $n < m$). (We write $\sim$ instead of $\overset{a}{\sim}$.) This $\sim$ is clearly reflexive and symmetric. To see transitivity let $m \sim n \,\&\, n \sim p$ (w.l.o.g $m < n < p$). Then $a \in E_{m,n} \,\&\, a \in E_{n,p}$. But $E_{m,n} \cap E_{n,p} \subseteq E_{m,p}$ so $m \sim p$.

It is trivial to show that for $i < j : h_i(a) = h_j(a)$ iff $a \in E_{i,j}$. That is $\{E_{i,j} : 1 \leq i < j \leq k\}$ is the equality set of R. $\square$

After Theorem 2.1 there is a natural way to axiomatize full families of dependencies of any type. This follows next:

F′-axiom. Let $F \subseteq P(\Omega) \times P(\Omega)$. Then F *satisfies the F'-axiom* if there is a natural number k and an indexed set of subsets of Ω, $[E_{i,j} : 1 \leq i < j \leq k\}$ such that

(i) If $(X,Y)\in P(\Omega)\times P(\Omega)\setminus F$, then there are $1\le i<j\le k$ such that $X\subseteq E_{i,j}$ and $Y\nsubseteq E_{i,j}$.

(ii) If $(X,Y)\in F$, $1\le i<j\le k$ and $X\subseteq E_{i,j}$, then $Y\subseteq E_{i,j}$.

(iii) For any $1\le i<j<l\le k$ $\{E_{i,j},\ E_{i,l},\ E_{j,l}\}$ is a Δ-system.

D′-axiom. Let $D\subseteq P(\Omega)\times P(\Omega)$. Then *D satisfies the D′-axiom* if there is a natural number k and an indexed set of subsets of Ω, $\{E_{i,j}:1\le i<j\le k\}$ such that

(i) If $(X,Y)\in P(\Omega)\times P(\Omega)\setminus D$, then there are $1\le i<j\le k$ such that $X\cap E_{i,j}\neq\emptyset$ and $Y\cap E_{i,j}=\emptyset$.

(ii) If $(X,Y)\in D$, $1\le i<j\le k$ and $X\cap E_{i,j}\neq\emptyset$, then $Y\cap E_{i,j}\neq\emptyset$.

(iii) The same as (iii) of the F′-axiom.

S′-axiom. Let $S\subseteq P(\Omega)\times P(\Omega)$. Then *S satisfies the S′-axiom* if there is a natural number k and an indexed set of subsets of Ω, $\{E_{i,j}:1\le i<j\le k\}$ such that

(i) If $(X,Y)\in P(\Omega)\times P(\Omega)\setminus S$, then there are $1\le i<j\le k$ such that $X\cap E_{i,j}\neq\emptyset$ and $Y\nsubseteq E_{i,j}$.

(ii) If $(X,Y)\in S$, $1\le i<j\le k$ and $X\cap E_{i,j}\neq\emptyset$, then $Y\subseteq E_{i,j}$.

(iii) The same as (iii) of the F′-axiom.

W′-axiom. Let $W\subseteq P(\Omega)\times P(\Omega)$. Then *W satisfies the W′-axiom* if there is a natural number k and an indexed set of subsets of Ω, $\{E_{i,j}:1\le i<j\le k\}$ such that

(i) If $(X,Y)\in P(\Omega)\times P(\Omega)\setminus W$, then there are $1\le i<j\le k$ such that $X\subseteq E_{i,j}$ and $Y\cap E_{i,j}=\emptyset$.

(ii) If $(X,Y)\in W$, $1\le i<j\le k$ and $X\subseteq E_{i,j}$, then $Y\cap E_{i,j}\neq\emptyset$.

(iii) The same as (iii) of the F′-axiom.

Remark. Observe that the $E_{i,j}$'s in the F′-axiom are maximal dependent sets, i.e. if $(X,Y)\in F$ and $X\subseteq E_{i,j}$, then $Y\subseteq E_{i,j}$.

Theorem 2.2. (i) *Let $Y\subseteq P(\Omega)\times P(\Omega)$ and $Y\in\{F,D,S\}$. Then Y satisfies the Y-axiom iff Y satisfies Y′-axiom.*

(ii) *Let Ω be a finite set, $|\Omega|\ge 3$. Then there is a $W\subseteq P(\Omega)\times P(\Omega)$ such that W satisfies the W-axiom and W doesn't satisfy the W′-axiom.*

Proof. (i) Let first $Y=F$ and suppose that Y satisfies the F-axiom. Write $Y=F$. For any $(X,Y)\in P(\Omega)\times P(\Omega)\setminus F$ take an $E(X,Y)\subseteq\Omega$ guaranteed by the F-axiom. List these $E(X,Y)$'s as $E_2,\dots,E_k$ (the indexes begin with 2!). For $1<j\le k$ let $E_{1,j}=E_j$ and for $1<i<j\le k$ let $E_{i,j}=E_i\cap E_j$. We claim that $\{E_{i,j}:1\le i<j\le k\}$ demonstrates that F satisfies the F′-axiom. The requirement (i) of the F′-axiom holds by $\{E_2,\dots,E_k\}\subseteq E_{i,j}:1\le i<j\le k\}$. We leave to the reader to check that (ii) holds too. To prove (iii) of the F′-axiom let $1\le i<j<l\le k$. We distinguish two cases:

(a) $i=1$. Then $E_{i,j}=E_j$; $E_{i,l}=E_l$ and $E_{i,l}=E_j\cap E_l$. Thus the intersection of two members of $\{E_{i,j};E_{i,l};E_{j,l}\}$ is $E_j\cap E_l$. This means that $\{E_{i,j};E_{i,l};E_{j,l}\}$ is a Δ-system.

(b) $1<i$. Then $E_{i,j}=E_i\cap E_j; E_{i,l}=E_i\cap E_l$ and $E_{j,l}=E_j\cap E_l$. Thus the intersection of any two members of $\{E_{i,j};E_{i,l};E_{j,l}\}$ is $E_i\cap E_j\cap E_l$. This means that $\{E_{i,j};E_{i,l};E_{j,l}\}$ is a Δ-system.

If Y satisfies the F′-axiom then Y obviously satisfies the F-axiom.

Now let $Y=D$ and suppose that Y satisfies the D-axiom. Write $Y=D$. For any $(X,Y)\in P(\Omega)\times P(\Omega)\setminus D$ take an $E(X,Y)\subseteq\Omega$ guaranteed by the D-axiom. List these $E(X,Y)$'s as $E_1,\dots E_k$. For $1\le i\le k$ let $E_{2i-1,2i}=E_i$ and if $1\le i<j\le 2k$ and $E_{i,j}$ is still undefined, then let $E_{i,j}=\emptyset$. It is easy to see that $\{E_{i,j}:1\le i<j\le 2k\}$ shows the D′-axiom to hold for D. If D satisfies the D′-axiom, then it trivially satisfies the D-axiom.

The case $Y=S$ is an easy modification of the proof worked in the case $Y=F$.

(ii) For the sake of simplicity suppose that $\Omega=\{a,b,c\}$. (In the general case pick two different elements of Ω, a,b. The role of $\{c\}$ will be played by $\Omega\setminus\{a,b\}$.) Let $W=\{(A,B)\in P(\Omega)\times P(\Omega)$: $A\subseteq\{a\}$ implies $a\in B$ and $A\subseteq\{b\}$ implies $b\in B\}$. Then W satisfies the W-axiom while if $(A,B)\in P(\Omega)\times P(\Omega)\setminus W$, then either ($A\subseteq\{a\}$ and $a\notin B$) or ($A\subseteq\{b\}$ and $b\notin B$). For (A,B), $E=\{a\}$ taken in the first case and $E=\{b\}$ in the second shows the W-axiom to hold.

We claim that W doesn't satisfy the W′-axiom. Suppose indirectly that $\mathscr{E}=\{E_{i,j}:1\le i<j\le k\}$ is a system that shows the W′-axiom to hold for W. Then

(1) $\{a\}\in\mathscr{E}$ and $\{b\}\in\mathscr{E}$ while $(\{a\},\Omega\setminus\{a\})\in P(\Omega)\times P(\Omega)\setminus W$ and $(\{b\},\Omega\setminus\{b\})\in P(\Omega)\times P(\Omega)\setminus W$ hold.

(2) $\emptyset\notin W$ and $\{c\}\notin W$ while $(\emptyset,\Omega)\in W$ and $(\{c\},\Omega\setminus\{c\})\in W$ hold.

By the 'allocation' of $\{a\}$ and $\{b\}$, we distinguish two cases:

(a) $\{a\}=E_{i,j}$ and $\{b\}=E_{i,l}$. Then $\{E_{i,j};E_{i,l};E_{j,l}\}$ is a Δ-system, that is $E_{j,l}=\emptyset$ or $E_{j,l}=\{c\}$ which contradicts (2).

(b) $\{a\}=E_{i,j}$ and $\{b\}=E_{l,m}$, where $|\{i,j,l,m\}|=4$. Now we are interested but in $E_{i,j}$; $E_{i,l}$; $E_{i,m}$; $E_{j,l}$; $E_{j,m}$ and $E_{l,m}$, thus we may suppose that $i=1$, $j=2$, $l=3$ and $m=4$.

Investigate what may be $E_{1,3}$. The cases $E_{1,3}=\{a\}$ or $\{b\}$ arise to (a). $E_{1,3}\neq\{c\}$ and $E_{1,3}\neq\emptyset$ by (2). $E_{1,3}\neq\{b,c\}$ while $\{E_{1,2};E_{1,3};E_{2,3}\}$ is a Δ-system hence $E_{1,3}=\{b,c\}$ implies $E_{2,3}=\emptyset$ contradicting (2). Now it is clear that $a\in E_{1,3}$. Thus $a\in E_{2,3}$, while $\{E_{1,2};E_{1,3};E_{2,3}\}$ is a Δ-system.

$\{E_{2,3};E_{2,4};E_{3,4}\}$ is a Δ-system, hence $a\notin E_{2,4}$, that is $E_{2,4}\subseteq\{b,c\}$. $E_{2,4}\neq\emptyset$ and $E_{2,4}\neq\{c\}$ by (2) and $E_{2,4}\neq\{b\}$ by (a). Hence $E_{2,4}=\{b,c\}$. $\{E_{2,3};E_{2,4};E_{3,4}\}$ is a Δ-system, hence $b\in E_{2,3}$.

Finally $E_{1,3}=\{a,c\}$ while $E_{2,3};E_{1,2};E_{1,3}$ form a Δ-system. Now $\{E_{1,3};E_{1,4};E_{3,4}\}$ is a Δ-system and $E_{1,3}\cap E_{3,4}=\emptyset$ and $E_{1,3}\cup E_{3,4}=\Omega$, hence $E_{1,4}=\emptyset$ which contradicts (2). $\square$

Remark. Theorem 2.2 demonstrates the difference between the weak dependency and the rest.

Theorem 2.3. *Let $Y\subseteq P(\Omega)\times P(\Omega)$ satisfy the Y'-axiom for some $Y\in\{F,D,S,W\}$.*

Then there is a relation R over Ω with $Y = Y_R$. Conversely if R is a relation over Ω then Y_R satisfies the Y'-axiom.

Proof. Let $\mathscr{E} = \{E_{i,j} : 1 \le i < j \le k\}$ show that Y satisfies the Y′-axiom. Then the requirement (iii) of the Y′-axiom and Theorem 2.1(ii) imply that there is a relation R over Ω such that $\mathscr{E}_R = \mathscr{E}$. By the Y′-axiom it is obvious that $Y = Y_R$. Conversely, if R is a relation over Ω, then writing $R = \{h_1, \dots, h_k\}$, $E_{i,j} = E(h_i, h_j)$; $\{E_{i,j} : 1 \le i < j \le k\}$ shows that Y_R satisfies the Y′-axiom. □

3. Combinatorial results

Definition 3.1. Let $\mathbf{F}$ be a full f-family and let $A \subseteq \Omega$. Then A is a *candidate key for* $\mathbf{F}$ if $(A, \Omega) \in \mathbf{F}$ and for any $A' \subset A$, $(A', \Omega) \notin \mathbf{F}$ holds. Let R be a relation over Ω, then the set of candidate keys of R is the set of candidate keys of $\mathbf{F}_R$.

Let $\mathbf{C}$ denote the set of candidate keys of $\mathbf{F}$. Then $\mathbf{C}$ is a Sperner system i.e. $(\forall A, B \in \mathbf{C})(A \subseteq B \rightarrow A = B)$.

We deal with the following question of [11]:

(∗) What is the largest number $r(n)$ of rows that is needed for some $\mathbf{C} \subseteq P(\Omega)$ being the set of candidate keys of a relation over Ω with $r(n)$ rows, where $|\Omega| = n$ and $\mathbf{C}$ is a Sperner system?

In [11] it is shown that for any Sperner system there is a relation with this system as its set of candidate keys and that

$$\sqrt{2\binom{n}{[\frac{1}{2}n]}} \le r(n) \le 2\binom{n}{[\frac{1}{2}n]}.$$

We give sharper estimations for $r(n)$.

Theorem 3.1.

$$\frac{1}{n^2}\binom{n}{[\frac{1}{2}n]} < r(n) \le \binom{n}{[\frac{1}{2}n]} + 1.$$

Proof. First we prove the upper bound. Let $\mathbf{C} \subseteq P(\Omega)$ be a Sperner system. Let $\mathbf{B}$ consist of the maximal sets that do not contain members of $\mathbf{C}$. Let the members of $\mathbf{B}$ be $B_2, \dots, B_k$. For $1 < j \le k$ let $E_{1,j} = B_j$ and for $1 < i < j \le k$ let $E_{i,j} = B_i \cap B_j$. Then $\{E_{i,j} : 1 \le i < j \le k\}$ satisfies the requirements of the Theorem 2.1(ii), hence there is a relation R over Ω with k rows such that $\mathscr{E}_R = \{E_{i,j} : 1 \le i < j \le k\}$. Then obviously $\mathbf{C}$ is the set of candidate keys of R. It is trivial that $\mathbf{B}$ is a Sperner system, and thus $|\mathbf{B}| \le \binom{n}{[n/2]}$ that is $k \le \binom{n}{[n/2]} + 1$.

Now let us see the lower bound. We start with two trivial observations.

(1) Let R be a relation over Ω with r rows. Then there is a relation R' over Ω such that R' uses no more than r symbols and $\mathscr{E}_R = \mathscr{E}_{R'}$.

(2) Let R be a relation over Ω with r rows and let $r' > r$. Then there is a relation R' over Ω with r' rows such that $\mathscr{E}_R = \mathscr{E}_{R'}$.

By (1) and (2) the number of Sperner systems which may be represented as sets of candidate keys of a relation with r rows is no more then $r^{r \cdot n}$. Hence

$$r(n)^{r(n) \cdot n} > 2^{\binom{n}{[n/2]}}$$

which implies $r(n) > \binom{n}{[n/2]}/n^2$. □

If $\mathbf{B}$ is a Sperner system and R is a relation such that $\mathbf{B} \subseteq \mathscr{E}_R \subseteq \{\bigcap \mathbf{B}' : \mathbf{B}' \subseteq \mathbf{B}\}$, then we can define two graphs on the set of rows of R as follows:

(1) The $\mathbf{B}$-graph of R is G_R where the vertices of G_R are the rows of R and two rows are connected by an edge if and only if their equality-set is an element of $\mathbf{B}$.

(2) The colored graph of R is the complete graph on the set of rows of R with the color $E(f, g)$ on the edge $\{f, g\}$.

The $\mathbf{B}$-graph of R has the following property: if G_R is disconnected, then there is a relation R' such that the number of rows of R' is less than that of R and $\mathbf{B} \subseteq \mathscr{E}_{R'} \subseteq \{\bigcap \mathbf{B}' : \mathbf{B}' \subseteq \mathbf{B}\}$.

The colored graph of R contains no circuit the edges of which have the same color except exactly one.

These two observations may be useful to make an algorithm to find the minimal relation for Sperner systems.

The estimation for $r(n)$ in Theorem 3.1 is not sharp. If $\mathbf{B} = \{X \subseteq \Omega : |X| = [\frac{1}{2}n]\}$, then there is a relation R such that $\mathbf{B} \subseteq \mathscr{E}_R \subseteq \{\bigcap \mathbf{B}' : \mathbf{B}' \subseteq \mathbf{B}\}$ and the number of rows of R is the least natural number greater than

$$\frac{1}{2}\binom{n}{[\frac{1}{2}n]} + \sqrt{2\binom{n}{[\frac{1}{2}n]}}.$$

It is natural to ask the following analogon of (*):

What is the largest number $R(n)$ of rows that is needed to represent a relation with $\mathbf{F}$ as the set of functional dependencies of it for an $\mathbf{F} \subseteq P(\Omega) \times P(\Omega)$ where $|\Omega| = n$ and $\mathbf{F}$ is a full f-family.

By the proof of Theorem 2.2(i) it is obvious that $R(n) \leq$ (the maximal number of subsets of Ω such that the intersection of any two of them is not a third). Thus, by a theorem of D. Kleitman [13], $R(n) \leq c \cdot \binom{n}{[n/2]}$ where $c = \frac{3}{2}$. Z. Füredi and J. Pach have shown, that this number is less than $(1 + (c \cdot \log n)/n)\binom{n}{[n/2]}$. It is trivial that $r(n) \leq R(n)$.

Lastly we give the combinatorial characterization - according to the Introduction - of the sets which are of minimal cardinality with respect to the property that they imply all the dependencies of a given full f-family.

We need some definitions and a lemma.

Definition 3.2. Let $\mathbf{M} \subseteq P(\Omega)$.

(i) We say that $\mathbf{M}$ *has the intersection property if for any* $\mathbf{M}' \subseteq \mathbf{M}$, $\bigcap \mathbf{M}' \in \mathbf{M}$ holds.

(ii) An $M \in \mathbf{M}$ *is irreducible if* $M \neq \bigcap \{M' \in \mathbf{M} : M \subset M'\}$ (recall that $\subset$ means strict inclusion).

(iii) An $\mathbf{N} \subseteq \mathbf{M}$ *generates* $\mathbf{M}$ if $\mathbf{M} = \{\bigcap \mathbf{N}' : \mathbf{N}' \subseteq \mathbf{N}\}$.

Lemma 3.1. *Let* $\mathbf{M}$ *have the intersection property and let* $\mathbf{N} = \{M \in \mathbf{M} : M$ *is irreducible*$\}$. *Then an* $\mathbf{N}' \subseteq \mathbf{M}$ *generates* $\mathbf{M}$ *iff* $\mathbf{N} \supseteq \mathbf{N}'$.

Proof. The following proof is standard in lattice theory. If $\mathbf{N}'$ generates $\mathbf{M}$, then $\mathbf{N} \subseteq \mathbf{N}'$ is obvious. For the converse we have to prove that $\mathbf{N}$ generates $\mathbf{M}$. Suppose indirectly that there is an $X \in \mathbf{M} \setminus \mathbf{N}$ such that $X \neq \bigcap \{Y : Y \in \mathbf{N} \,\&\, X \subseteq Y\}$. Let X be of minimal cardinality with respect to this property. $X \notin \mathbf{N}$ means that $X = \bigcap \{Y : Y \in \mathbf{M} \,\&\, X \subset Y\}$, hence $X \subset Y$ implies that there is an $\mathbf{N}_Y \subseteq \mathbf{N}$ such that $Y = \bigcap \mathbf{N}_Y$. Let $\mathbf{N}_X = \bigcup \{\mathbf{N}_Y : X \subset Y$ and $Y \in \mathbf{M}\}$.

Then $\mathbf{N}_X \subseteq \mathbf{N}$ and $X = \bigcap \mathbf{N}_X$ which is a contradiction. □

Remark. Observe that the proof of Theorems in [2] are essentially our proof of Lemma 3.1.

Corollary. *If* $\mathbf{M}$ *has the intersection property, then there is exactly one* $\mathbf{N} \subseteq \mathbf{M}$ *which generates* $\mathbf{M}$ *and has minimal cardinality.*

Theorem 3.2. *Let* $\mathbf{F}$ *be a full f-family, let* $\mathbf{B}$ *be the set of maximal dependent set for* $\mathbf{F}$ *and let* $\mathbf{C}$ *be the set which generates* $\mathbf{B}$ *and has minimal cardinality* (*in* [1] *there is shown that* $\mathbf{B}$ *has the intersection property*). *Then for any* $\mathbf{F}' \subseteq \mathbf{F}$ *we have the following:*

$\mathbf{F}'$ *implies all the dependencies of* $\mathbf{F}$ *and* $\mathbf{F}'$ *has minimal cardinality with respect to this property, if and only if, for any* $C \in \mathbf{C}$ *there is an* $A_C \subseteq \Omega$ *such that* $\mathbf{F}' = \{(A_C, C) : C \in \mathbf{C}\}$.

We leave the easy proof of this theorem to the reader. We think that it is interesting to compare Theorem 3.2 with the theorem on p. 16 of [2].

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